

# ALTERNATING REACHABILITY

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*Dedicated to the memory of Malka Peled*

**ABSTRACT.** We consider a graph with colored edges. A trail (vertices may repeat but not edges) is called *alternating* when successive edges have different colors. Given a set of vertices called *terminals*, the *alternating reachability* problem is to find an alternating trail connecting distinct terminals, if one exists. A special case with two colors is searching for an augmenting path with respect to a given matching. In another special case with two colors red and blue, the *alternating cone* is defined as the set of assignments of nonnegative weights to the edges such that at each vertex, the total red weight equals the total blue weight; in a companion paper we showed how the search for an integral weight vector within a given box in the alternating cone can be reduced to the alternating reachability problem in a 2-colored graph. We define an obstacle, called a *Tutte set*, to the existence of an alternating trail connecting distinct terminals in a colored graph, and give a polynomial-time algorithm, generalizing the blossom algorithm of Edmonds, that finds either an alternating trail connecting distinct terminals or a Tutte set. We use Tutte sets to show that an edge-colored bridgeless graph where each vertex has incident edges of at least two different colors has a closed alternating trail. A special case with two colors one of which forms a matching yields a combinatorial result of Giles and Seymour. We show that in a 2-colored graph, the cone generated by the characteristic vectors of closed alternating trails is the intersection of the alternating cone with the cone generated by the characteristic vectors of cycles in the underlying graph.

## 1. INTRODUCTION AND SUMMARY

Let  $G = (V, E)$  be a graph (we allow parallel edges but not loops). A *walk* in  $G$  is a sequence

$$W = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m), \quad m \geq 0,$$

where  $v_i \in V$  for all  $i$ ,  $e_j \in E$  for all  $j$ , and  $e_j$  has endpoints  $v_{j-1}$  and  $v_j$  for all  $j$ . We say that  $v_1, v_2, \dots, v_{m-1}$  are the *internal vertices* of the walk  $W$ . Note that since we are allowing repetitions, the vertices  $v_0, v_m$  could also be internal vertices. The walk  $W$  is said to be *closed* when  $v_0 = v_m$  and is said to be a *trail* when the edges  $e_1, \dots, e_m$  are distinct.

Now assume that the edges of  $G$  are colored with a set  $C$  of colors, where  $\#C \geq 2$ , the coloring being given by  $\mathcal{C} : E \rightarrow C$ . We say that  $(G, \mathcal{C})$  is an *edge-colored graph*. The walk  $W$  above is *internally alternating* when  $\mathcal{C}(e_j) \neq \mathcal{C}(e_{j+1})$  for each  $j = 1, \dots, m-1$ , and is *alternating* when in addition if  $W$  is closed then  $\mathcal{C}(e_m) \neq \mathcal{C}(e_1)$  (note that a walk can be closed and internally alternating without being alternating, but if  $v_0 \neq v_m$ , there is no distinction between internally alternating and alternating walks and we use the word alternating in this case). A closed alternating walk (respectively, trail) is abbreviated as *CAW* (respectively, *CAT*).

Given an edge-colored graph  $G = (V, E)$  and a set  $S \subseteq V$  of vertices called *terminals*, the *alternating reachability problem* is to either find an alternating trail connecting distinct terminals or show that none exists. Our motivation for considering this problem arises from the following two special cases with two colors, say red and blue.

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(a) Let  $G = (V, E)$  be a simple graph and let  $M \subseteq E$  be a matching. Color the edges in  $M$  red and the edges in  $E - M$  blue, and let  $S$  be the set of exposed vertices of  $M$ . The alternating reachability problem for these data is equivalent to finding an augmenting path with respect to  $M$ , if one exists.

(b) Let  $G = (V, E)$  be a graph whose edges have been colored red and blue. The *alternating cone* is defined as the set of assignments of nonnegative weights to the edges such that at each vertex, the total red weight equals the total blue weight (we define the alternating cone more formally below). In a companion paper [BPS1] we showed how the search for an integral weight vector within a given box in the alternating cone can be reduced to the alternating reachability problem in a 2-colored graph.

We now outline the main results of this paper.

In Section 2 we show, generalizing the approach of Edmonds [E] as explained in Lovász and Plummer [LP], that given an edge-colored graph  $G = (V, E)$  and a set of terminals, either there is an alternating trail connecting distinct terminals, or else there is a subset of nonterminals, called a *Tutte set*, that acts as an obstruction to such alternating trails (for a precise definition of a Tutte set, see Section 2). Moreover, we present a polynomial-time algorithm that finds one or the other. The alternating reachability problem (in a slightly different version) was first considered by Tutte, in a nonalgorithmic form, in his book on graph theory [T1] and in the paper [T2]. Tutte [T2] called the obstructions to alternating trails connecting distinct terminals  $r$ -barriers. Every  $r$ -barrier is a Tutte set but not conversely. It turns out that there is a very minor error in Tutte's work: Tutte's theory actually produces Tutte sets in our sense and not  $r$ -barriers in his sense. Also, it is easy to give an example where there exists a unique Tutte set, which is not an  $r$ -barrier.

In Section 3 we use Tutte sets to prove a combinatorial result on CAT's in edge-colored bridgeless graphs. To motivate this result consider the following statements:

- (i) Let  $D$  be a directed graph in which every vertex has positive indegree and outdegree. Then  $D$  has a directed circuit (this is easily proved).
- (ii) Let  $(G, \mathcal{C})$  be an edge-colored graph such that for every vertex  $v$  of  $G$ , we can find edges of two different colors incident with  $v$ . Then  $(G, \mathcal{C})$  has a CAW (this can be proved by a simple alternating walk argument similar to the proof of Theorem 2.2 of [BPS1]).

Theorem 3.1 strengthens the hypothesis of statement (ii) above by assuming in addition that  $G$  is bridgeless. It concludes, using Tutte sets, that  $(G, \mathcal{C})$  has a CAT. We also deduce a result (Theorem 3.4) due to Giles and Seymour [S] on cycles in bridgeless graphs from a special case of Theorem 3.1, where there are two colors, red and blue, and the red edges form a matching.

In Section 4 we use Theorem 3.1 to prove an intersection theorem for certain polyhedral cones associated to *2-colored graphs* (i.e., edge-colored graphs where the number of colors is two) that were defined in [BPS1]. We now recall some definitions from [BPS1].

Let  $G = (V, E)$  be a graph. Assume that the edges of  $G$  are colored red or blue, the coloring being given by  $\mathcal{C} : E \rightarrow \{R, B\}$ . Consider the real vector space  $\mathbb{R}^E$ , with coordinates indexed by the set of edges of  $G$ . We write an element  $x \in \mathbb{R}^E$  as  $x = (x(e) : e \in E)$ . For an edge  $e \in E$ , the characteristic vector  $\chi(e) \in \mathbb{R}^E$  is defined by  $\chi(e)(f) = \begin{cases} 1, & \text{if } f = e \\ 0, & \text{if } f \neq e. \end{cases}$

The *cone of closed alternating walks* or simply the *alternating cone*  $\mathcal{A}(G, \mathcal{C})$  of a 2-colored graph  $(G, \mathcal{C})$  (when the coloring  $\mathcal{C}$  is understood, we suppress it from the notation and write  $\mathcal{A}(G)$ ), is defined to be the set of all vectors  $x = (x(e) : e \in E)$  in  $\mathbb{R}^E$  satisfying the following system of homogeneous linear inequalities:

$$(1) \quad \sum_{e \in E_R(v)} x(e) - \sum_{e \in E_B(v)} x(e) = 0, \quad v \in V,$$

$$(2) \quad x(e) \geq 0, \quad e \in E.$$

We refer to (1) as the *balance condition* at vertex  $v$ . Figure 1 illustrates a 2-colored graph together with an integral vector in its alternating cone. In [BPS1] we determined the extreme rays and

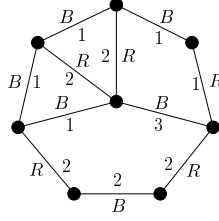


FIGURE 1. An integral vector in the alternating cone

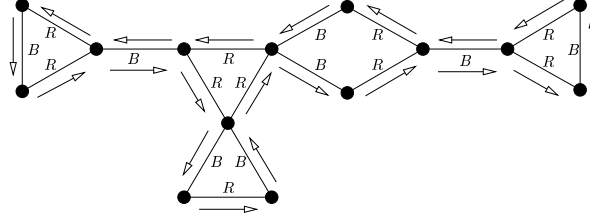


FIGURE 2. An irreducible CAW

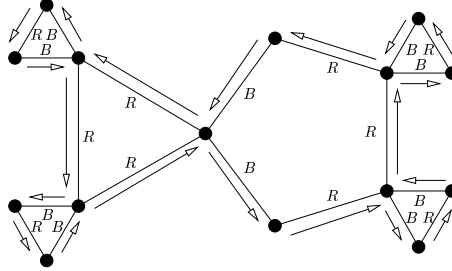


FIGURE 3. An irreducible CAT

dimension of the alternating cone and showed that searching for an integral vector within a given box in the alternating cone can be reduced to searching for an alternating trail connecting two given vertices in a residual 2-colored graph.

The characteristic vector of a walk  $W = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m)$  is defined to be  $\chi(W) = \sum_{i=1}^m \chi(e_i)$ . A CAW  $W$  in a 2-colored graph is said to be *irreducible* if  $\chi(W)$  cannot be written as  $\chi(W_1) + \chi(W_2)$  for any CAW's  $W_1$  and  $W_2$ . Similarly, a CAT  $T$  is said to be *irreducible* if  $\chi(T)$  cannot be written as  $\chi(T_1) + \chi(T_2)$  for any CAT's  $T_1$  and  $T_2$ . Figure 2 depicts an irreducible CAW (with direction of walk indicated by an arrow) and Figure 3 depicts an irreducible CAT. Irreducibility is easily seen. A simple alternating walk argument (see [BPS1]) shows that every integral vector in the alternating cone is a nonnegative integral combination of characteristic vectors of irreducible CAW's.

Circulations in directed graphs can be thought of in terms of flows along the arcs obeying the conservation constraint at every vertex. For example, the characteristic vector of a directed circuit corresponds to a unit of flow along the circuit. Such an interpretation is not available in the case of vectors in the alternating cone. The irreducible CAW of Figure 2 does not correspond to a flow in an intuitive sense. On the other hand, the characteristic vector of an irreducible CAT *can* be thought of as a unit of flow around the trail. For a 2-colored graph  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$ , it is thus natural to consider the convex polyhedral cone  $\mathcal{T}(G, \mathcal{C}) \subseteq \mathbb{R}^E$  generated by the characteristic

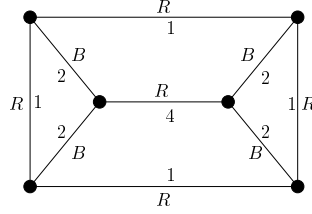


FIGURE 4. A vector in the alternating cone but not in the trail cone

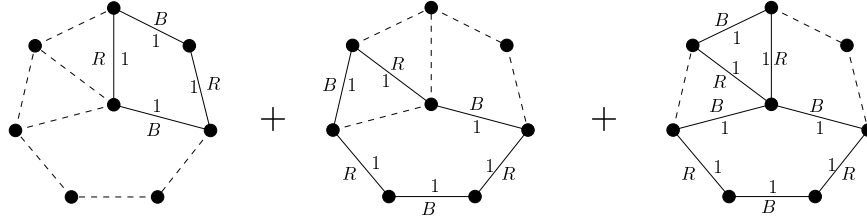


FIGURE 5. CAT's whose sum is the vector of Figure 1

vectors of the CAT's in  $(G, \mathcal{C})$ . We call  $\mathcal{T}(G, \mathcal{C})$  the *cone of closed alternating trails*, or simply the *trail cone*, of  $(G, \mathcal{C})$ . For example, it is easily seen that the integral vector in the alternating cone from Figure 4 is not in the trail cone. On the other hand, the integral vector in the alternating cone from Figure 1 can be written as a sum of characteristic vectors of three CAT's, as shown in Figure 5.

Consider a CAT in a 2-colored graph. Its characteristic vector satisfies the balance condition at every vertex. If we ignore the colors, the edge set of the CAT is a disjoint union of the edge sets of some cycles in the underlying graph. This shows that a nonnegative integral combination (that is to say, a linear combination with nonnegative integral *coefficients*) of characteristic vectors of CAT's satisfies the balance condition at every vertex and can be written as a nonnegative integral combination of characteristic vectors of cycles in the underlying graph. We conjecture that the converse of this observation is also true:

**Conjecture 1.1.** *Let  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$  be a 2-colored graph and let  $y \in \mathbb{N}^E$ . Then  $y$  is a nonnegative integral combination of characteristic vectors of CAT's in  $(G, \mathcal{C})$  if and only if*

- (i)  *$y$  satisfies the balance condition at every vertex, i.e.,  $y \in \mathcal{A}(G, \mathcal{C})$ ;*
- (ii)  *$y$  can be written as a nonnegative integral combination of characteristic vectors of cycles in  $G$ .*

Let  $\mathcal{Z}(G)$  denote the cone in  $\mathbb{R}^E$  generated by the characteristic vectors of the cycles in  $G$ . Seymour [S] found the linear inequalities determining  $\mathcal{Z}(G)$  (Theorem 4.1). The observation in the paragraph preceding Conjecture 1.1 shows that  $\mathcal{T}(G, \mathcal{C}) \subseteq \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ . From Conjecture 1.1 it is easy to show that  $\mathcal{T}(G, \mathcal{C}) = \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ : take a rational vector  $y \in \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ ; for a suitably large positive integer  $k$ ,  $ky$  satisfies conditions (i) and (ii) of the conjecture; by the conjecture  $ky$  (respectively,  $y$ ) is a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of CAT's. In Section 4 we prove that indeed  $\mathcal{T}(G, \mathcal{C}) = \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ . Our proof is an adaptation of Seymour's argument for  $\mathcal{Z}(G)$ . Seymour's inductive proof is based on the Giles-Seymour lemma (Theorem 3.4) and likewise, our inductive proof uses Theorem 3.1.

We remark that in this paper we focus on graph-theoretical aspects of the alternating cone and not on algorithmic efficiency. We do consider algorithms, but always with a view to obtaining graph-theoretical results.

We now collect in one place certain commonly used definitions in the rest of the paper. Let  $G = (V, E)$  be a graph and consider a walk

$$(3) \quad W = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m), \quad m \geq 0,$$

in  $G$ . We say that  $W$  is a  $v_0$ - $v_m$  walk of length  $m$ . We call  $e_1$  the *first* edge of  $W$  and  $e_m$  the *last* edge of  $W$ . The walk  $W^R$  is the  $v_m$ - $v_0$  walk obtained by reversing the sequence (3). The walk  $W$  is said to be

*a path* when the edges  $e_1, \dots, e_m$  are distinct and the vertices  $v_0, \dots, v_m$  are distinct;  
*a cycle* when  $W$  is closed, the edges  $e_1, \dots, e_m$  are distinct, and the vertices  $v_0, \dots, v_{m-1}$  are distinct.

We have defined paths and cycles as special classes of walks. However, sometimes it is more convenient to think of paths and cycles as subgraphs, as is done usually. This will be clear from the context. If  $W_1$  is a  $u$ - $v$  walk and  $W_2$  is a  $v$ - $w$  walk, then the *concatenation* of  $W_1$  and  $W_2$ , denoted  $W_1 * W_2$ , is the  $u$ - $w$  walk obtained by walking from  $u$  to  $v$  along  $W_1$  and continuing by walking from  $v$  to  $w$  along  $W_2$ . Note that if  $W_1$  and  $W_2$  are trails, then  $W_1 * W_2$  is a trail whenever  $W_1$  and  $W_2$  have no edges in common.

Now let  $(G, \mathcal{C})$  be a 2-colored graph. The walk  $W$  in (3) is said to be

*an even alternating cycle*  
 when  $W$  is a cycle of even length and  $W$  is alternating; an even alternating cycle will also be called simply an *alternating cycle*;  
*an odd internally alternating cycle with base  $v_0$*   
 when  $W$  is a  $v_0$ - $v_0$  cycle of odd length and  $W$  is internally alternating.

## 2. ALTERNATING TRAILS IN AN EDGE-COLORED GRAPH

Let  $G = (V, E)$  be a graph and let  $\mathcal{C} : E \rightarrow C$  be an edge coloring. In this section we consider the *alternating reachability problem*: given a set  $S$  of vertices called *terminals*, either find an alternating trail connecting distinct terminals or show that no such trail exists. For the rest of this section we consider  $G, \mathcal{C}$  and  $S$  as fixed.

The problem of finding a CAT through a given edge  $e$  in an edge-colored graph can be easily reduced to the alternating reachability problem: let  $e = \{s, t\}$ ,  $s \neq t$ . Remove  $e$  from the graph, add two new vertices  $s'$  and  $t'$ , add a new edge with the color  $\mathcal{C}(e)$  between  $s'$  and  $s$  and one between  $t'$  and  $t$ , and let  $S = \{s', t'\}$ . Clearly the alternating reachability problem in the new graph is equivalent to the original problem.

As mentioned in the introduction, the problem of finding an augmenting path with respect to a given matching is also reducible to the alternating reachability problem.

The alternating reachability problem was first considered by Tutte [T1, T2]. We discuss Tutte's work at the end of this section. Our solution to the alternating reachability problem is along the lines of the blossom forest algorithm of Edmonds [E], as explained in Section 9.1 of Lovász and Plummer's book [LP]. The solution is in terms of Tutte sets, defined below.

**Definition 2.1.** A subset  $A \subseteq (V - S)$  is a *Tutte set* when

- (i) each component of  $G - A$  has at most one terminal;
- (ii)  $A$  can be written as a disjoint union (denoted  $\dot{\cup}$ , empty blocks allowed)

$$A = \dot{\bigcup}_{c \in C} A(c)$$

such that conditions (a), (b), and (c) below hold.

A vertex  $u \in A$  is said to have *color*  $c$  if  $u \in A(c)$  (there is a unique such  $c$ ). An edge  $e \in E$  is said to be *mismatched* if  $e$  connects a vertex  $u \in A$  with a vertex  $v \in V - A$  and  $\mathcal{C}(e)$  is different from the color of  $u$ , or  $e$  connects two vertices  $u, v \in A$  and  $\mathcal{C}(e)$  is different from both the colors of  $u$  and of  $v$ .

Conditions (a), (b), and (c) are as follows:

- (a) if  $H$  is a component of  $G - A$  containing a terminal, then there is no mismatched edge with an endpoint in  $H$ ;
- (b) if  $H$  is a component of  $G - A$  containing no terminals, then there is at most one mismatched edge with an endpoint in  $H$ ;
- (c) there are no mismatched edges with both endpoints in  $A$ .

The next theorem shows that a Tutte set is an obstruction to the existence of an alternating trail connecting distinct terminals.

**Theorem 2.2.** *Suppose a Tutte set  $A$  exists. Let  $s$  be a terminal, let  $H$  be a component of  $G - A$  containing the vertex  $t$  but not  $s$ , and assume that there is no mismatched edge with an endpoint in  $H$ . Then there is no alternating  $s$ - $t$  trail. In particular, there is no alternating trail connecting distinct terminals.*

*Proof.* We assert that if an alternating trail  $T$  enters  $A$  from  $V - A$  via an edge that is not mismatched, then the next time  $T$  leaves  $A$ , it can only be via a mismatched edge. Specifically, suppose that

$$T = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m)$$

is an alternating trail with  $v_0 \in V - A$ ,  $v_1 \in A$ , and  $\mathcal{C}(e_1) = \text{color of } v_1$ . Assume that there exists some  $j \geq 2$  with  $v_j \in V - A$ , and let  $i$  be the least such  $j$ . Proving the assertion amounts to showing  $\mathcal{C}(e_i) \neq \text{color of } v_{i-1}$ .

We show by induction that all  $l = 1, \dots, i-1$  satisfy  $\mathcal{C}(e_l) = \text{color of } v_l$ . The base case  $l = 1$  follows from hypothesis. Now assume that the statement holds for  $l = t$ , where  $t < i-1$ . Thus  $\mathcal{C}(e_t) = c$ , where  $c$  is the color of  $v_t$ . Since  $T$  is alternating,  $d = \mathcal{C}(e_{t+1}) \neq c$ . Since  $v_{t+1} \in A$ , it follows from condition (ii)(c) in Definition 2.1 that the color of  $v_{t+1}$  is  $d$ .

We have shown that  $\mathcal{C}(e_{i-1}) = \text{color of } v_{i-1}$ . Since  $T$  is alternating,  $\mathcal{C}(e_i) \neq \mathcal{C}(e_{i-1})$ , and thus  $\mathcal{C}(e_i) \neq \text{color of } v_{i-1}$ , which proves the assertion.

Now suppose that  $T$  is an alternating  $s$ - $t$  trail. Since  $s$  and  $t$  are in different components of  $G - A$ ,  $T$  must enter  $A$ . By (ii)(a) in Definition 2.1, the first time  $T$  enters  $A$ , it must be via an edge that is not mismatched. Since  $t \notin A$ ,  $T$  must leave  $A$ . By the assertion, the first time  $T$  leaves  $A$ , it is via a mismatched edge. By hypothesis, the component  $K$  of  $G - A$  that  $T$  enters upon leaving  $A$  (for the first time) cannot be the destination component  $H$  containing  $t$ , and thus  $T$  must leave  $K$  and enter  $A$  again. By (ii)(b) in Definition 2.1, this entry must be via an edge that is not mismatched (since the only mismatched edge has already been used for entering  $K$ ). Therefore, by the assertion, when  $T$  leaves  $A$  for the next time, it must be via a mismatched edge. Continuing this argument we see that every time  $T$  leaves  $A$ , it must be via a mismatched edge. Thus  $T$  can never reach the destination component  $H$  containing  $t$ , a contradiction.  $\square$

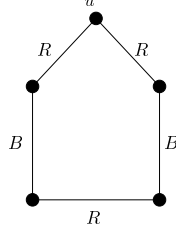
For later use we record the following lemma.

**Lemma 2.3.** *Suppose a Tutte set  $A$  exists. If  $s$  is a terminal and  $u \in A(c)$ , then the last edge of each alternating  $s$ - $u$  trail has color  $c$ .*

*Proof.* Let

$$T = (v_0, e_1, v_1, e_2, v_2, \dots, e_m, v_m)$$

be an alternating trail with  $s = v_0$  and  $u = v_m$ . By (ii)(a) in Definition 2.1, we see that when  $T$  enters  $A$  for the first time, it must be via an edge that is not mismatched. Just as in Lemma 2.2, we can now show that every time  $T$  leaves  $A$ , it must be via a mismatched edge, and every time  $T$  enters  $A$ , it must be via an edge that is not mismatched. Since  $u \in A$ ,  $T$  must enter  $A$  for the last time, say via the edge  $e_l$ . Since  $e_l$  is not mismatched,  $\mathcal{C}(e_l) = \text{color of } v_l$ . The induction argument used in proving the assertion in the proof of Theorem 2.2 now shows that  $\mathcal{C}(e_m) = \text{color of } v_m = c$ .  $\square$


 FIGURE 6. A  $B$ -blossom with base  $u$ 

We now want to show that the converse of Theorem 2.2 holds, i.e., if there is no alternating trail connecting distinct terminals, then a Tutte set exists. We shall present an algorithm that finds such a trail or a Tutte set. First, we need to make a few definitions.

For a non root vertex  $u$  in a rooted forest, the *predecessor edge* of  $u$  is the first edge in the unique path from  $u$  to the root of the component containing  $u$ . Given a partition  $\pi$  of the vertex set of  $G$ , by the *shrunk graph*  $G \times \pi$  we mean the graph obtained from  $G$  by shrinking each block of  $\pi$  into a vertex and discarding loops. In other words, the vertex set of  $G \times \pi$  is the set of blocks of  $\pi$ , the edge set of  $G \times \pi$  is the set of edges of  $G$  whose endpoints lie in different blocks of  $\pi$ , and the endpoints of an edge  $e$  (in  $G \times \pi$ ) are the blocks of  $\pi$  in which the endpoints of  $e$  (in  $G$ ) lie. For  $v \in V$ , the block of  $\pi$  containing  $v$  is denoted by  $[v]$ . For a subset  $U \subseteq V$ , the subgraph of  $G$  induced on  $U$  is denoted by  $G[U]$ .

**Definition 2.4.** A subgraph  $G' = (V', E')$  of  $G$  is said to be a *blossom with base  $u$*  when

- (i)  $u \in V'$ ;
- (ii) for each  $v \in V'$  such that  $v \neq u$ ,  $G'$  has two  $v$ - $u$  alternating trails whose first edges have different colors.

Condition (ii) allows us to extend any alternating trail reaching  $G'$  up to  $u$ .

Note that a subgraph consisting of a single vertex  $u$  is a blossom with base  $u$ .

**Definition 2.5.** Let  $c \in C$ . A subgraph  $G' = (V', E')$  of  $G$  is said to be a  *$c$ -blossom with base  $u$*  when

- (i)  $u \in V'$ ;
- (ii) for each  $v \in V'$  such that  $v \neq u$ ,  $G'$  has two  $v$ - $u$  alternating trails whose first edges have different colors and whose last edges have colors different from  $c$ ;
- (iii)  $G'$  has a  $u$ - $u$  internally alternating trail of positive length whose first and last edges have colors different from  $c$ .

Conditions (ii) and (iii) allow us to extend any alternating trail reaching  $G'$  (even a trail reaching  $u$  via an edge of color  $c$ ) up to  $u$  and then be ready to continue with color  $c$ .

Note that for all  $c$ , a subgraph consisting of a single vertex  $u$  is *not* a  $c$ -blossom.

**Example 2.6.** The following are examples of blossoms.

- (i) If there are only two colors  $R$  and  $B$ , an odd internally alternating  $u$ - $u$  cycle with two  $R$  edges incident at  $u$  is a  $B$ -blossom (see Figure 6).
- (ii) Assuming there are three colors  $R$ ,  $B$  and  $G$ , Figure 7 depicts two  $B$ -blossoms with base  $u$ .
- (iii) Figure 8 depicts two more  $B$ -blossoms with base  $u$ . At this point this can be verified directly from Definition 2.5. Later we shall see that the first of these  $B$ -blossoms arises by shrinking and the second by fusion.

**Definition 2.7.** By a *colored blossom forest* we mean a triple  $(I, \pi, F)$ , where  $S \subseteq I \subseteq V$ ,  $\pi$  is a partition of  $I$ , and  $F$  is a rooted forest in  $G[I] \times \pi$ , satisfying the following conditions:



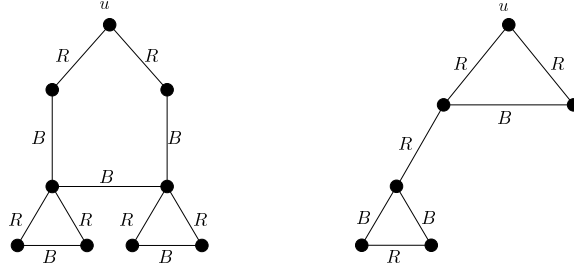
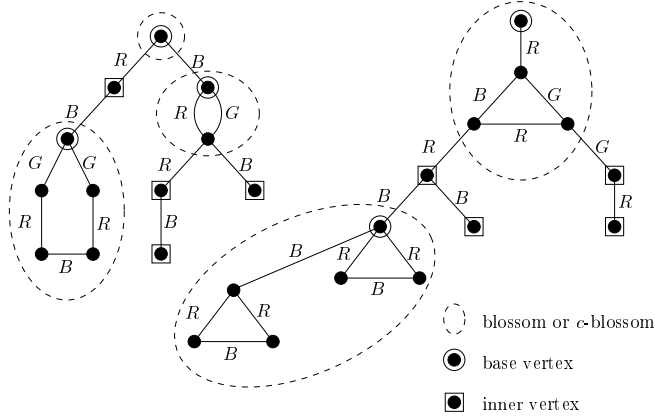
FIGURE 7. Two  $B$ -blossoms with base  $u$ FIGURE 8.  $B$ -blossoms with base  $u$  obtained by shrinking and by fusion

FIGURE 9. A colored blossom forest

- (i)  $F$  has  $\#S$  components and the roots of  $F$  are  $[s]$ ,  $s \in S$ ;
- (ii) for each terminal  $s \in S$ , the induced subgraph  $G[[s]]$  is a blossom with base  $s$ ;
- (iii) let  $[v]$  be an non root vertex of  $F$  satisfying  $\#[v] = 1$ , let  $e$  be the predecessor edge of  $[v]$  in  $F$ , and let  $e_1, e_2, \dots, e_k$  be the edges between  $[v]$  and its children in  $F$ ; then  $e_1, \dots, e_k$  all have colors different from  $\mathcal{C}(e)$ ;
- (iv) let  $[v]$  be an non root vertex of  $F$  satisfying  $\#[v] \geq 2$ , let the predecessor edge of  $[v]$  in  $F$  have color  $c$ , and let its endpoint (in  $G$ ) that is contained in  $[v]$  be  $u \in [v]$ ; then the induced subgraph  $G[[v]]$  is a  $c$ -blossom with base  $u$ .

Figure 9 depicts a colored blossom forest, where the blocks of  $\pi$  are the blossoms,  $c$ -blossoms, and singleton inner vertices indicated in the figure.

A colored blossom forest always exists: set  $I = S$ , take  $\pi$  to be the trivial partition of  $S$  whose blocks are the singletons, and take  $F$  to be the rooted forest with no edges and roots  $[s]$ ,  $s \in S$ ; then  $(S, \pi, F)$  is a colored blossom forest.



Given a colored blossom forest  $(I, \pi, F)$ , we classify the vertices of  $G$  as follows. Vertices in  $V - I$  are called *out-of-forest vertices*. For  $v \in I$ , if  $\# [v] = 1$  and  $[v]$  is a non root vertex of  $F$ , then we call  $v$  an *inner vertex*. All other vertices in  $I$  are called *blossom vertices*. If  $v$  is an inner or blossom vertex in  $V$ , we also call  $[v]$  an inner or blossom vertex in  $F$ , respectively. In item (ii) of Definition 2.7, we say that  $s$  is the *base* of the blossom vertex  $[s]$  of  $F$ , and in item (iv) of Definition 2.7, we say that  $u$  is the *base* of the blossom vertex  $[v]$  of  $F$ .

We now define certain positive-length internally alternating trails in  $G$  w.r.t. a colored blossom forest  $(I, \pi, F)$ .

Let  $u$  be a blossom vertex such that  $[u]$  is not a root of  $F$ . Let the predecessor edge of  $[u]$  in  $F$  have color  $c$  and assume that  $u$  is the base of the blossom vertex  $[u]$  of  $F$ . Pick an internally alternating  $u$ - $u$  trail of positive length in  $G[[u]]$  whose first and last edges have colors different from  $c$  (such a trail is guaranteed by the definition of a colored blossom forest), and call it  $T(u, F)$ .

Let  $u$  be a blossom vertex such that  $[u]$  is not a root of  $F$ . Let the predecessor edge of  $[u]$  in  $F$  have color  $c$  and now assume that  $v, v \neq u$ , is the base of the blossom vertex  $[u]$  of  $F$ . Pick two alternating  $u$ - $v$  trails in  $G[[u]]$  whose first edges have different colors and whose last edges have colors different from  $c$  (such trails are guaranteed by the definition of a colored blossom forest), and call them  $T_1(u, F)$  and  $T_2(u, F)$ .

Let  $u \notin S$  be a blossom vertex such that  $[u]$  is a root of  $F$ . Let  $s \in S$  be the base of the blossom vertex  $[u]$  of  $F$ . Pick two alternating  $u - s$  trails contained in  $G[[s]]$ , whose first edges have different colors (such trails are guaranteed by the definition of a colored blossom forest), and call them  $T_1(u, F)$  and  $T_2(u, F)$ .

The next lemma defines certain internally alternating trails in  $G$ , and states their properties. Some of these trails may be of length zero and some may be equal. These conventions prevent some case distinctions later on.

**Lemma 2.8.** *Let  $(I, \pi, F)$  be a colored blossom forest, and let  $u, v$  be vertices of  $G$  satisfying the following conditions:*

- $u$  and  $v$  both belong to  $I$ ;
- $[u]$  is a descendent of  $[v]$  in  $F$ ;
- $v$  is either an inner vertex or the base of the blossom vertex  $[v]$  of  $F$ .

*Then  $G$  has internally alternating trails  $T_1(u, v, F)$ ,  $T_2(u, v, F)$  satisfying the following properties.*

- (i) *If  $u$  is an inner vertex, then  $T_1(u, v, F) = T_2(u, v, F)$ .*
- (ii) *The trail  $T_1(u, v, F)$  is of length zero precisely when  $v = u$ . The trail  $T_2(u, v, F)$  is of length zero precisely when  $u = v$  is inner or  $u = v = s$  for some  $s \in S$ .*
- (iii) *The edges of the trails  $T_1(u, v, F)$  and  $T_2(u, v, F)$  include all the edges in the  $[u]$ - $[v]$  path in  $F$ .*
- (iv) *If an edge  $e$  in  $T_1(u, v, F)$  or  $T_2(u, v, F)$  is not in  $F$ , then both its endpoints in  $G$  are contained in a blossom vertex of  $F$  lying on the  $[u]$ - $[v]$  path in  $F$ .*
- (v) *If  $u$  is an inner vertex and  $v \neq u$ , then the first edge of  $T_1(u, v, F)$  is the predecessor edge of  $[u]$  in  $F$ .*
- (vi) *If  $v$  is an inner vertex and  $u \neq v$ , then the last edges of  $T_1(u, v, F)$  and  $T_2(u, v, F)$  have colors different from the color of the predecessor edge of  $[v]$  in  $F$ .*
- (vii) *If  $u$  is a blossom vertex and  $v \neq u$ , then the first edges of  $T_1(u, v, F)$ , and  $T_2(u, v, F)$  have different colors.*
- (viii) *If  $v \notin S$  is the base of a blossom vertex, then the first and last edges of  $T_2(v, v, F)$  have colors different from the color of the predecessor edge of  $[v]$  in  $F$ .*
- (ix) *If  $v \notin S$  is the base of a blossom vertex and  $u \neq v$ , then the last edges of  $T_1(u, v, F)$  and  $T_2(u, v, F)$  have colors different from the color of the predecessor edge of  $[v]$  in  $F$ .*

*Proof.* The proof is by induction on the distance  $d([u], [v])$  in  $F$  between the vertices  $[u]$  and  $[v]$ .

First assume that  $d([u], [v]) = 0$ , i.e.,  $u$  and  $v$  belong to the same block of  $\pi$ . The following cases arise.

**Case (a):**  $u$  and  $v$  are both inner. Define  $T_1(u, v, F) = T_2(u, v, F) = (u)$ , the zero-length trail starting and ending at  $u$ .

**Case (b):**  $u$  and  $v$  are both blossom vertices and  $u \neq v$ . Define

$$T_1(u, v, F) = T_1(u, F), \quad T_2(u, v, F) = T_2(u, F).$$

**Case (c):**  $u$  and  $v$  are both blossom vertices and  $u = v$ ,  $u \notin S$ . Define

$$T_1(u, v, F) = (u), \quad T_2(u, v, F) = T(u, F).$$

**Case (d):**  $u$  and  $v$  are both blossom vertices and  $u = v$ ,  $u \in S$ . Define

$$T_1(u, v, F) = T_2(u, v, F) = (u).$$

It is easily seen that  $T_1(u, v, F)$  and  $T_2(u, v, F)$  are internally alternating and conditions (i)–(ix) in the statement of the lemma are satisfied (when restricted to  $u, v$  satisfying  $d([u], [v]) = 0$ ).

Now assume that  $d([u], [v]) > 0$ . Then  $u \neq v$ , and let  $e$  be the predecessor edge of  $[u]$  in  $F$ . Let the endpoints of  $e$  in  $G$  be  $x$  and  $y$ , where  $x \in [u]$ . Then  $d([y], [v]) = d([u], [v]) - 1$  and  $T_1(y, v, F)$  and  $T_2(y, v, F)$  will have been defined. The following cases arise.

**Case (a):**  $u$  is inner. We have  $u = x$ . By induction and (vii) and (iii) of Definition 2.7, one of

$$(u, e, y) * T_1(y, v, F), \quad (u, e, y) * T_2(y, v, F)$$

is alternating; define  $T_1(u, v, F) = T_2(u, v, F)$  to be that trail (breaking ties arbitrarily).

**Case (b):**  $u$  is a blossom vertex and  $u \neq x$ . By the definition of  $T_1(u, F)$ , induction and (vii), one of

$$T_1(u, F) * (x, e, y) * T_1(y, v, F), \quad T_1(u, F) * (x, e, y) * T_2(y, v, F)$$

is alternating; define  $T_1(u, v, F)$  to be that trail (breaking ties arbitrarily). Similarly, one of

$$T_2(u, F) * (x, e, y) * T_1(y, v, F), \quad T_2(u, F) * (x, e, y) * T_2(y, v, F)$$

is alternating; define  $T_2(u, v, F)$  to be that trail (breaking ties arbitrarily).

**Case (c):**  $u$  is a blossom vertex and  $u = x$ ,  $u \notin S$ . By (iii) of Definition 2.7, induction and (vii), one of

$$(x, e, y) * T_1(y, v, F), \quad (x, e, y) * T_2(y, v, F)$$

is alternating; define  $T_1(u, v, F)$  to be that trail (breaking ties arbitrarily). Similarly and by the definition of  $T(u, F)$ , one of

$$T(u, F) * (x, e, y) * T_1(y, v, F), \quad T(u, F) * (x, e, y) * T_2(y, v, F)$$

is alternating; define  $T_2(u, v, F)$  to be that trail (breaking ties arbitrarily).

**Case (d):**  $u$  is a blossom vertex and  $u = x$ ,  $u \in S$ . This implies  $u = v$  and thus  $d([u], [v]) = 0$ . So this case cannot occur.

It is easily checked that  $T_1(u, v, F)$  and  $T_2(u, v, F)$  are alternating and conditions (i)–(ix) in the statement of the lemma are satisfied.  $\square$

Let  $(I, \pi, F)$  be a colored blossom forest and let  $e$  be an edge in  $G$ , with endpoints  $u$  and  $v$  belonging to  $I$ , that is not an edge of  $F$ . Assume that

- $[u]$  and  $[v]$  are not in the same component of  $F$ .
- If  $[u]$  is an inner vertex of  $F$ , then  $\mathcal{C}(e)$  is different from the color of the predecessor edge of  $[u]$  in  $F$ .
- If  $[v]$  is an inner vertex of  $F$ , then  $\mathcal{C}(e)$  is different from the color of the predecessor edge of  $[v]$  in  $F$ .

In this situation we say that we have a *breakthrough*.

**Lemma 2.9.** *Assume we have a breakthrough, with the notation as in the preceding paragraph. Let  $[s], [t]$ ,  $s, t \in S$ ,  $s \neq t$  be the roots of the components of  $F$  containing  $[u]$  and  $[v]$ , respectively. Then  $G$  has an alternating  $s$ - $t$  trail.*

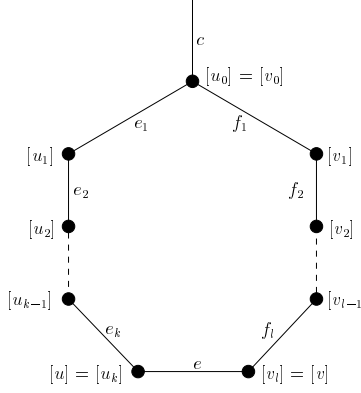


FIGURE 10. Illustrating a shrinking

*Proof.* By (vii) of Lemma 2.8 and the definition of breakthrough, for some  $p, q \in \{1, 2\}$ ,

$$T_p(u, s, F)^R * (u, e, v) * T_q(v, t, F)$$

is an alternating  $s$ - $t$  trail in  $G$ . □

We now discuss three operations on a colored blossom forest  $(I, \pi, F)$ : growing, shrinking and fusing. The operation of fusing does not occur in the classical case of searching for augmenting paths in a nonbipartite graph.

Let  $e$  be an edge of  $G$  between vertices  $u$  and  $v$  that is not an edge of  $F$ .

Assume that

- $u \in I, v \notin I$ .
- If  $[u]$  is an inner vertex of  $F$ , then  $\mathcal{C}(e)$  is different from the color of the predecessor edge of  $[u]$  in  $F$ .

Add the singleton block  $\{v\}$  to  $\pi$  to get a partition  $\pi'$  of  $I' = I \cup \{v\}$ . Let  $F'$  denote the rooted forest in  $G \times \pi'$  obtained by adding the inner vertex  $\{v\}$  to the vertices of  $F$  and adding the edge  $e$  to the set of edges of  $F$ . It is easily seen that  $(I', \pi', F')$  is a colored blossom forest. We say that  $(I', \pi', F')$  is obtained from  $(I, \pi, F)$  by *growing*.

We now define the operation of shrinking.

Let  $e$  be an edge of  $G$  between vertices  $u$  and  $v$  that is not an edge of  $F$ .

Assume that

- $u, v \in I$ .
- $[u]$  and  $[v]$  are in the same component of  $F$  and  $[u] \neq [v]$ .
- If  $[u]$  is an inner vertex of  $F$ , then  $\mathcal{C}(e)$  is different from the color of the predecessor edge of  $[u]$  in  $F$ .
- If  $[v]$  is an inner vertex of  $F$ , then  $\mathcal{C}(e)$  is different from the color of the predecessor edge of  $[v]$  in  $F$ .

Adding the edge  $e$  to  $F$  creates a unique cycle  $K$ . Let  $[u_0] = [v_0]$  denote the unique common ancestor in  $F$  of  $[u]$  and  $[v]$  that belongs to  $K$ . Denote the vertices on the  $[u_0]$ - $[u]$  path in  $F$  by  $[u_0], [u_1], \dots, [u_{k-1}], [u_k] = [u]$ , and denote the edge of  $F$  between  $[u_{i-1}]$  and  $[u_i]$  by  $e_i$ . Similarly, denote the vertices on the  $[v_0]$ - $[v]$  path in  $F$  by  $[v_0], [v_1], \dots, [v_{l-1}], [v_l] = [v]$ , and denote the edge of  $F$  between  $[v_{i-1}]$  and  $[v_i]$  by  $f_i$ . See Figure 10. Note that  $k$  or  $l$  may be zero but  $k + l \geq 1$ .

Replace the blocks  $[u_0], \dots, [u_k], [v_0], \dots, [v_l]$  of  $\pi$  by their union to obtain a partition  $\pi'$  of  $I$  with  $\pi < \pi'$ . Define a rooted forest  $F'$  in  $G \times \pi'$  by throwing away the edges  $e_1, \dots, e_k, f_1, \dots, f_l$  from  $F$

(the remaining edges in  $F$  have obvious endpoints in  $G \times \pi'$ ). For convenience, we denote the block of  $\pi$  containing  $w \in I$  by  $[w]$  and the block of  $\pi'$  containing  $w$  by  $\overline{w}$ .

We assert that  $(I, \pi', F')$  is a colored blossom forest. Clearly, conditions (i) and (iii) in Definition 2.7 are satisfied by  $(I, \pi', F')$ , and we need to check condition (ii) or (iv) only for the vertex  $\overline{u_0}$  (according as  $[u_0]$  is a root of  $F$  or not). We assume that  $[u_0]$  is not a root of  $F$  and check condition (iv); the case when  $[u_0]$  is a root of  $F$  and we need to check condition (ii) is similar, and we omit it.

Without loss of generality we may assume that if  $[u_0]$  is a blossom vertex of  $F$ , then its base is  $u_0$ . Let  $c$  be the color of the predecessor edge of  $[u_0]$  in  $F$ .

We show that  $G[\overline{u_0}]$  is a  $c$ -blossom with base  $u_0$ . First we check condition (ii) of Definition 2.5. Let  $w \in \overline{u_0}$ ,  $w \neq u_0$ . Without loss of generality we may assume that  $w \in [u_i]$  for some  $i = 0, \dots, k$ .

The following two cases arise.

**Case (a):**  $[u_i]$  is a blossom vertex of  $F$ . By (vii) and (ix) of Lemma 2.8, we have that  $T_1(w, u_0, F)$  and  $T_2(w, u_0, F)$  have first edges of different colors and have last edges with colors different from  $c$ .

**Case (b):**  $[u_i]$  is an inner vertex of  $F$ . In this case  $w = u_i$ . From the assumption on  $[u]$  and  $[v]$  and (vii) of Lemma 2.8, it follows that for some  $p, q \in \{1, 2\}$ , the  $w$ - $u_0$  trail

$$T' = T_p(u, w, F)^R * (u, e, v) * T_q(v, u_0, F)$$

is alternating. Since  $w$  is an inner vertex, it now follows from (v) and (vi) of Lemma 2.8 that  $T_1(w, u_0, F)$  and  $T'$  have first edges of different colors, and by (ix) of Lemma 2.8 their last edges have colors different from  $c$ .

We have verified condition (ii) in Definition 2.5. Now consider condition (iii). As before, for some  $p, q \in \{1, 2\}$ , the  $u_0$ - $u_0$  trail

$$T_p(u, u_0, F)^R * (u, e, v) * T_q(v, u_0)$$

of positive length is internally alternating with first and last edges of colors different from  $c$ .

This completes the proof of the assertion that  $(I, \pi', F')$  is a colored blossom forest. We say that  $(I, \pi', F')$  is obtained from  $(I, \pi, F)$  by *shrinking*.

Now we define the operation of fusion. Let  $e$  be an edge of  $G$  between vertices  $u$  and  $v$  that is an edge of  $F$ . Assume that

- $[u]$  and  $[v]$  are (necessarily distinct) blossom vertices of  $F$ .

Then  $e$  must be the predecessor edge of one of  $[u]$  or  $[v]$ . Replace the blocks  $[u]$  and  $[v]$  of  $\pi$  by their union to obtain a partition  $\pi'$  of  $I$ . Define a rooted forest  $F'$  in  $G \times \pi'$  by throwing away the edge  $e$  from  $F$  (the remaining edges in  $F$  have obvious endpoints in  $G \times \pi'$ ). Again we denote the block of  $\pi'$  containing a vertex  $w \in I$  by  $\overline{w}$ .

We assert that  $(I, \pi', F')$  is a colored blossom forest. Without loss of generality, assume that  $e$  is the predecessor edge of  $[u]$ . Clearly, conditions (i) and (iii) in Definition 2.7 are satisfied by  $(I, \pi', F')$ , and we need to check condition (ii) or (iv) only for the vertex  $\overline{v}$  (according as  $[v]$  is a root of  $F$  or not). We assume that  $[v]$  is not a root of  $F$  and check condition (iv); the case when  $[v]$  is a root of  $F$  and we need to check condition (ii) is similar, and we omit it.

Let  $x$  be the base of the blossom vertex  $[v]$  of  $F$ , and let  $c$  be the color of the predecessor edge of  $[v]$  in  $F$ . We show that  $G[\overline{x}]$  is a  $c$ -blossom with base  $x$ . Condition (iii) in Definition 2.5 clearly holds, since  $G[[x]]$  is contained in  $G[\overline{x}]$ . Condition (ii) of Definition 2.5 follows from Lemma 2.8: given that  $w \in \overline{x}$ ,  $w \neq x$ , the two alternating  $w$ - $x$  trails  $T_1(w, x, F)$  and  $T_2(w, x, F)$  are contained in  $G[\overline{x}]$ , have first edges of different colors, and have last edges of colors different from  $c$ .

This proves the assertion that  $(I, \pi', F')$  is a colored blossom forest. We say that  $(I', \pi', F')$  is obtained from  $(I, \pi, F)$  by *fusion*.

We now put a partial order on colored blossom forests. Given colored blossom forests  $\alpha = (I, \pi, F)$  and  $\beta = (I', \pi', F')$ , we say that  $\alpha < \beta$  if  $I$  is a proper subset of  $I'$ , or  $I = I'$  and  $\pi < \pi'$  (as partitions, i.e., every block of  $\pi$  is contained in a block of  $\pi'$  and  $\pi \neq \pi'$ ).

The next theorem is the promised converse of Theorem 2.2.

**Theorem 2.10.** *Suppose that  $G$  has no alternating trail connecting distinct terminals. Let  $(I, \pi, F)$  be a maximal colored blossom forest. Let  $A$  be the set of inner vertices of  $G$  with respect to  $(I, \pi, F)$ . For each  $c \in C$ , define*

$$A(c) = \{u \in A : \text{the predecessor edge of } [u] \text{ in } F \text{ has color } c\}.$$

*Then  $A$  is a Tutte set with coloring given by  $A = \dot{\cup}_{c \in C} A(c)$ .*

Before proving Theorem 2.10, we prove the following properties of the maximal colored blossom forest  $(I, \pi, F)$ .

**Lemma 2.11.** *Under the assumptions of Theorem 2.10:*

- (i) *No edge in  $G$  connects a blossom vertex and an out-of-forest vertex.*
- (ii) *If an edge  $e$  in  $G$  connects an inner vertex and an out-of-forest vertex, then  $C(e)$  agrees with the color of the inner vertex.*
- (iii) *No edge in  $G$  connects blossom vertices contained in two different vertices of  $F$ .*
- (iv) *If an edge  $e$  in  $G$  connects two inner vertices, then  $C(e)$  agrees with the color of one of them.*
- (v) *If an edge  $e$  in  $G$  connects a blossom vertex  $v$  and an inner vertex  $u$ , then  $C(e)$  agrees with the color of  $u$ , except when  $v$  is the base of a blossom and  $e$  is the predecessor edge of  $[v]$  in  $F$ , in which case the colors are different.*

*Proof.* If condition (i) or (ii) does not hold, we can grow the colored blossom forest  $(I, \pi, F)$ , contradicting its maximality.

We now consider condition (iii). Let  $e$  be an edge in  $G$  between blossom vertices  $u$  and  $v$ . If  $[u]$  and  $[v]$  are in different components of  $F$ , then we have a breakthrough which, by Lemma 2.9, contradicts our assumption that there are no alternating trails connecting distinct terminals. If  $[u]$  and  $[v]$  are in the same component of  $F$  and  $e$  is an edge of  $F$ , then we can fuse, contradicting the maximality of  $(I, \pi, F)$ . If  $[u]$  and  $[v]$  are in the same component of  $F$  and  $e$  is not an edge of  $F$ , then we can shrink, again contradicting the maximality of  $(I, \pi, F)$ .

Now we verify condition (iv). Let  $e$  be an edge in  $G$  between the inner vertices  $u$  and  $v$ . If  $e$  is an edge of  $F$ , then it is the predecessor edge of one of  $[u]$  and  $[v]$ , and thus  $C(e)$  agrees with the color of  $u$  or  $v$ . If  $e$  is not an edge of  $F$  and  $[u]$  and  $[v]$  are in different components of  $F$  and  $C(e)$  is different from the colors of  $u$  and  $v$ , then we have a breakthrough, a contradiction. If  $e$  is not an edge of  $F$  and  $[u]$  and  $[v]$  are in the same component of  $F$  and  $C(e)$  is different from the colors of  $u$  and  $v$ , then we can shrink, a contradiction.

Finally, consider condition (v). Let  $e$  be an edge in  $G$  between a blossom vertex  $v$  and an inner vertex  $u$ . If  $e$  is an edge of  $F$ , then either  $e$  is the predecessor edge of  $[u]$ , in which case  $C(e)$  agrees with the color of  $u$ , or  $v$  is the base of the blossom vertex  $[v]$  of  $F$  and  $e$  is the predecessor edge of  $[v]$ , in which case  $C(e)$  is different from the color of  $u$ . If  $e$  is not an edge of  $F$ , then  $C(e)$  agrees with the color of  $u$ , because otherwise we have a breakthrough (if  $[v]$  and  $[u]$  are in different components of  $F$ ) or we can shrink (if  $[v]$  and  $[u]$  are in the same component of  $F$ ).  $\square$

*Proof of Theorem 2.10.* Let  $K_1, K_2, \dots, K_p$  be the components of the subgraph of  $G$  induced by the out-of-forest vertices. Write  $S = \{s_1, s_2, \dots, s_k\}$  and list the non inner vertices of  $F$  as

$$[s_1], [s_2], \dots, [s_k], [s_{k+1}], \dots, [s_l]$$

(so that the set of blossom vertices in  $V$  is precisely  $[s_1] \cup \dots \cup [s_l]$ ). Statements (i) and (iii) in Lemma 2.11 imply that the components of  $G - A$  are precisely

$$G[[s_1]], \dots, G[[s_l]], K_1, \dots, K_p.$$

Condition (i) in Definition 2.1 now follows. Statement (iv) in Lemma 2.11 proves condition (ii)(c) in Definition 2.1.

We verify condition (ii)(a) in Definition 2.1 using statement (v) in Lemma 2.11, by noting that  $[s_1], \dots, [s_k]$  are roots of  $F$  and therefore have no predecessor edge.

Condition (ii)(b) in Definition 2.1 follows from statements (ii) and (v) in Lemma 2.11: from (ii) we see that there are no mismatched edges with an endpoint in  $K_1, \dots, K_p$ , and from (v) we see that there is exactly one mismatched edge with an endpoint in each of  $G[[s_{k+1}]], \dots, G[[s_l]]$ .  $\square$

In the spirit of the Gallai-Edmonds decomposition, the out-of-forest, inner, and blossom vertices w.r.t. a maximal colored blossom forest can be characterized as follows.

Define  $N(S)$  to be the set of all vertices  $t \in V - S$  such that for all  $s \in S$ , there is no alternating  $s$ - $t$  trail in  $G$ .

Define  $T(S)$  to be the set of all vertices  $t \in V - S$  such that for some  $s \in S$ , there are two alternating  $s$ - $t$  trails whose last edges have different colors.

For a color  $c \in C$ , define  $I(S, c)$  to be the set of all vertices  $t \in V - S$  satisfying the following property: there are alternating trails starting from  $S$  and ending in  $t$ , and the last edges of all such alternating trails have the color  $c$ . Set  $I(S) = \dot{\cup}_{c \in C} I(S, c)$ .

Lemma 2.3 implies that if  $A$  is a Tutte set, then  $A \subseteq I(S) \cup N(S)$ . Theorem 2.12 below and Theorems 2.2 and 2.10 show that if a Tutte set exists, then  $I(S)$  is one (indeed, if a Tutte set exists, Theorem 2.2 shows that  $G$  has no alternating trail connecting distinct terminals; then Theorem 2.10 shows that the inner vertices of a maximal colored blossom forest form a Tutte set; and Theorem 2.12 shows that  $I(S)$  are the inner vertices).

**Theorem 2.12.** *Suppose that  $G$  has no alternating trail connecting distinct terminals. Let  $(I, \pi, F)$  be a maximal colored blossom forest. Then*

$$\begin{aligned} N(S) &= \text{Out-of-Forest Vertices} \\ I(S, c) &= \text{Inner Vertices with Predecessor Edges having Color } c \\ I(S) &= \text{Inner Vertices} \\ S \cup T(S) &= \text{Blossom Vertices} \end{aligned}$$

*Proof.* Let  $X$  be the set of out-of-forest vertices, and let  $Y$  be the set of blossom vertices. Define  $A$  and  $A(c)$  as in Theorem 2.10. We first show that  $X \subseteq N(S)$ ,  $A(c) \subseteq I(S, c)$ , and  $Y \subseteq S \cup T(S)$ .

Write the components of  $G - A$  (in the notation of the proof of Theorem 2.10) as

$$G[[s_1]], \dots, G[[s_l]], K_1, \dots, K_p.$$

By Lemma 2.11(ii), there is no mismatched edge with an endpoint in  $K_1, \dots, K_p$ . It now follows from Theorem 2.2 that there is no alternating trail from any vertex in  $S$  to any out-of-forest vertex, i.e.,  $X \subseteq N(S)$ .

For each  $u \in A(c)$ , it follow by Lemmas 2.8(v) and 2.3 that  $u \in I(S, c)$ . Thus  $A(c) \subseteq I(S, c)$ .

Lemma 2.8(vii) implies that  $Y \subseteq S \cup T(S)$ .

Since  $X, \dot{\cup}_{c \in C} A(c), Y$  partition  $V$  and  $N(S), I(S), S \cup T(S)$  are disjoint, it follows that  $X = N(S)$ ,  $A(c) = I(S, c)$ ,  $Y = S \cup T(S)$ .  $\square$

**Remark 2.13.** The theory presented in this section produces in polynomial time either an alternating trail connecting distinct terminals or a Tutte set. Indeed, start with the trivial colored blossom forest  $\pi$  defined after Definition 2.7, and perform growing, shrinking and fusing operations in any order until a breakthrough or a maximal colored blossom forest is achieved. Discovering that one of these operations is possible and performing it or discovering a breakthrough takes polynomial time. As for the number of operations, initially  $\pi$  has  $\#S$  blocks. Shrinking and fusing decrease the number of blocks of  $\pi$  and keep the number of out-of-forest vertices constant. Growing increases the first by one and decreases the second by one, so at most  $\#V - \#S$  growing steps can occur in total. It follows that termination must occur within  $\#V$  operations.

**Remark 2.14.** We now comment on Tutte's work on the alternating reachability problem. Tutte gives a nonalgorithmic solution to a slightly different version of the alternating reachability problem. Tutte calls the obstructions to the existence of alternating trails  $r$ -barriers [T2, page 331]. There is a

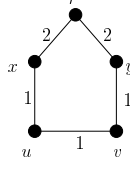


FIGURE 11. A counter-example to Theorem 5.1 of [T2]

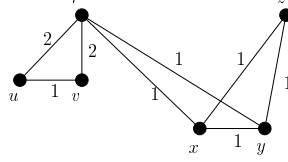


FIGURE 12. A counter-example to Theorem 5.3 of [T2]

small but important difference between our definition of Tutte set and the definition of  $r$ -barrier. If we were to apply Tutte's definition of  $r$ -barrier to the version of alternating reachability considered in this paper, then condition (c) in Definition 2.2(ii) would read:

(c) If an edge  $e$  connects two vertices of  $A$ , then these vertices have different colors and one of them has the color  $\mathcal{C}(e)$ ,

instead of our condition paraphrased:

(c) If an edge  $e$  connects two vertices of  $A$ , then one of them has the color  $\mathcal{C}(e)$ .

Thus every  $r$ -barrier is a Tutte set but not conversely. It is easy to find instances of the alternating reachability problem where there are no alternating trails connecting distinct terminals, but all obstructions are Tutte sets and not  $r$ -barriers. We give such an example for Tutte's original version of the alternating reachability problem.

We follow the definitions and notation of Tutte's paper [T2, page 325–326] without reproducing them here. Consider the graph in Figure 11 (1 and 2 are colors, as in Tutte's paper). In this graph there are no bicursal edges and the only acursal edge is between  $u$  and  $v$ , all vertices are uncursal, and  $U_1 = \{r, u, v\}$ ,  $U_2 = \{x, y\}$ . This is a counter-example to Tutte's Theorem 5.1, since  $(U_1, U_2)$  is not an  $r$ -barrier: the edge between  $u$  and  $v$  violates condition (ii) in the definition of an  $r$ -barrier.

Now consider the graph in Figure 12. It is easy to convince oneself of the following:

- (i) There is no “coloured path” in  $J(r)$  from  $r$  to  $z$ .
- (ii) There is no  $r$ -barrier such that  $z$  is a vertex of an inaccessible outer component.

In other words, with Tutte's definition of an  $r$ -barrier, his main result Theorem 5.3 is false. The solution is to replace condition (ii) in Tutte's definition of an  $r$ -barrier by the following:

- (ii) If both ends of an edge  $A$  of  $G$  are inner vertices of  $\chi$ , then one of them has the color of  $A$ .

This is precisely our notion of a Tutte set. We believe that the only (very minor) error in [T2] occurs in the proof of Theorem 5.1, and that the theory presented in that paper actually yields a Tutte set rather than an  $r$ -barrier. Tutte has several applications of this theorem in his book on graph theory [T1]. It is unlikely that these applications need the stronger notion of barriers; in most applications, Tutte sets would do. Edges like the one between  $u$  and  $v$  in the first example and between  $x$  and  $y$  in the second are acursal in Tutte's terminology, and therefore will not be used by  $J(r)$ .



### 3. CLOSED ALTERNATING TRAILS IN EDGE-COLORED BRIDGELESS GRAPHS

In this section we give an application of Tutte sets to closed alternating trails in edge-colored bridgeless graphs. This result will be used in the next section where we determine the inequalities defining the trail cone.

**Theorem 3.1.** *Let  $G = (V, E)$  be a bridgeless graph with an edge-coloring  $\mathcal{C} : E \rightarrow C$ . Assume that for each  $v \in V$ , there are two edges with different colors incident at  $v$ . Then  $(G, \mathcal{C})$  has a CAT.*

*Proof.* Suppose that  $(G, \mathcal{C})$  has no CAT. Let  $e$  be an edge in  $G$  between  $s$  and  $t$ . Form a new edge-colored graph  $G'$  from  $G$  by removing  $e$ , and adding two new vertices  $s'$  and  $t'$  and two new edges  $f$  between  $s'$  and  $s$  and  $g$  between  $t'$  and  $t$ , both with the color  $\mathcal{C}(e)$ . Since  $G$  has no CAT, it follows that  $G'$  has no  $s'$ - $t'$  alternating trail. Consider the alternating reachability problem for  $G'$  with  $S = \{s', t'\}$  and consider the sets  $I(S, c)$ ,  $c \in C$  defined in the previous section. Let  $A$  denote the set of inner vertices of a maximal colored blossom forest. It follows from Theorem 2.12 that  $A = I(S) = \dot{\cup}_{c \in C} I(S, c)$  is a Tutte set with  $s'$  and  $t'$  in different components of  $G' - A$  that are incident with no mismatched edges. Since  $e$  is not a bridge in  $G$ ,  $s'$  and  $t'$  are in the same component of  $G'$ , and hence  $A \neq \emptyset$ .

We first make the following two observations.

- (i) There are edges of two different colors incident at every vertex of  $G'$  other than  $s'$  and  $t'$ .
- (ii) If  $s \notin A$ , then  $s$  must be in the component of  $G' - A$  containing  $s'$ ; and similarly if  $t \notin A$ , then  $t$  must be in the component of  $G' - A$  containing  $t'$ .

By definition of  $I(S)$ , there exist alternating trails in  $G'$  starting at  $s'$  or  $t'$  and ending at a vertex in  $A$ . Let  $T$  denote the longest such alternating trail, starting at  $s'$  without loss of generality and ending at  $z \in A$ . Let  $d$  be the last edge of  $T$ . By Lemma 2.3,  $z \in I(S, \mathcal{C}(d))$ . By observation (i) above we can find an edge  $h$  of  $G'$  incident with  $z$  satisfying  $\mathcal{C}(d) \neq \mathcal{C}(h)$ . Let  $w$  be the other endpoint of  $h$ . We shall now derive a contradiction. The following three cases arise.

**Case (a):**  $T$  contains  $h$ . Since  $\mathcal{C}(d) \neq \mathcal{C}(h)$ , it follows from  $z \in I(S, \mathcal{C}(d))$  that  $T$  must have traversed  $h$  in the direction from  $z$  to  $w$ . Thus, the portion of  $T$  starting with  $h$  and ending with  $d$  is a CAT in  $G'$ . Since  $s'$  and  $t'$  have degree 1 in  $G'$ , this CAT cannot contain the edges  $f$  or  $g$ , so is a CAT in  $G$ , a contradiction.

**Case (b):**  $w \in A$  and  $T$  does not contain  $h$ . In this case,  $T * (z, h, w)$  is an alternating trail that is longer than  $T$  and ends at a vertex in  $A$ , a contradiction.

**Case (c):**  $w \notin A$  and  $T$  does not contain  $h$ . Let  $H$  be the component of  $G' - A$  containing  $w$ . Since  $\mathcal{C}(d) \neq \mathcal{C}(h)$ , it follows that  $h$  is mismatched and  $H$  does not contain  $s'$  or  $t'$ . Hence by observation (ii) above,  $s \in A$  or  $s \notin H$ , and similarly for  $t$ , so  $e$  does not connect  $A$  and  $H$ . By the proof of Lemma 2.3,  $T$  can enter a component of  $G' - A$  only via a mismatched edge. It follows (since  $h$  is the only mismatched edge incident with  $H$ ) that  $T$  never enters  $H$ , and so  $T$  has no vertices in  $H$ . Since  $z$  is an inner vertex and  $w$  is a blossom vertex, it follows from Lemma 2.11(v) that  $w$  is the base of a  $\mathcal{C}(h)$ -blossom, which coincides with  $H$  as in the proof of Theorem 2.10. If  $h$  were the only edge of  $G'$  between  $A$  and  $H$ , then it would follow that  $h$  is a bridge not only in  $G'$  but also in  $G$ , a contradiction. Thus  $G$  has some edge  $b$ ,  $b \neq h$  between  $x \in A$  and  $y \in H$ . Since  $T$  has no vertices in  $H$ , it contains neither  $h$  nor  $b$  nor any edge of  $H$ . By Lemma 2.8(ii), (vii), (ix) we see that one of  $T * (z, h, w) * T_1^R(y, w) * (y, b, x)$  and  $T * (z, h, w) * T_2^R(y, w) * (y, b, x)$  is an alternating trail from  $s'$  to  $A$  that is longer than  $T$ , a contradiction.  $\square$

**Remark 3.2.** The algorithm of Section 2 gives a polynomial-time method of finding a CAT in an edge-colored graph satisfying the hypothesis of Theorem 3.1: for each edge  $e$  of  $G$ , construct the corresponding graph  $G'$  and look for an alternating  $s'$ - $t'$  trail. For some edge  $e$  we will find an alternating  $s'$ - $t'$  trail in  $G'$ , which easily yields a CAT in  $G$ . Actually, the proof of Theorem 3.1 gives a method of finding a CAT in  $G$  by solving the alternating reachability problem (with  $S = \{s', t'\}$ ) for a single  $G'$  (for an arbitrary edge  $e$  in  $G$ ). We do not give the details.

**Remark 3.3.** Conjecture 1.1 can be seen as strengthening both the hypothesis and conclusion of Theorem 3.1 (in the case of 2 colors). The nontrivial part of the conjecture states that if a sum of cycles is balanced, then it is a sum of CAT's. We may delete bridges and then isolated vertices from  $G = (V, E)$ , since they are irrelevant to the conjecture. Then some positive vector  $y \in \mathbb{N}^E$  is a sum of cycles. Theorem 3.1 assumes that  $E$  (the support of  $y$ ) has edges of both colors at every vertex; Conjecture 1.1 assumes more, namely that  $y$  is balanced. Theorem 3.1 concludes that  $G$  has a CAT, whereas Conjecture 1.1 concludes more, namely that  $y$  is a sum of CAT's.

We conclude this section by deducing the lemma of Giles and Seymour on cycles in bridgeless graphs (see [S]) from Theorem 3.1.

**Theorem 3.4** (Giles and Seymour). *Let  $G = (V, E)$  be a bridgeless graph and let  $\phi : V \rightarrow E$  map each vertex  $v$  to an edge incident at  $v$ . Then  $G$  has a cycle  $C$  such that for each vertex  $w$  of  $C$ ,  $\phi(w)$  is an edge of  $C$ .*

*Proof.* Partition  $E$  as  $E = E_0 \dot{\cup} E_1 \dot{\cup} E_2$ , where for  $i = 0, 1, 2$ ,  $E_i$  consists of those edges  $e$  such that  $e = \phi(v)$  for exactly  $i$  endpoints  $v$  of  $e$ . Define a 2-colored graph as follows. First subdivide each edge in  $E_1$  by introducing a new vertex, i.e., for each edge  $e \in E_1$  between  $u$  and  $w$ , introduce a new vertex  $v_e$  and replace  $e$  with the two edges between  $u$  to  $v_e$  and between  $w$  to  $v_e$ . There is an obvious map  $\Sigma$  from the edges of the resulting graph  $G'$  onto  $E$  ( $\Sigma$  is the identity on  $E_0$  and  $E_2$  and takes each subdivided edge onto its parent edge in  $E_1$ ). We color the edges in  $E_0$  blue and the edges in  $E_2$  red. Now consider an edge  $e \in E_1$  between  $u$  and  $w$ , say with  $\phi(u) = e$  and  $\phi(w) \neq e$ . Then we color the edge between  $u$  and  $v_e$  red and the edge between  $w$  and  $v_e$  blue.

It is easily seen that

- (i)  $G'$  is bridgeless.
- (ii) Every vertex of  $G'$  has positive red and blue degrees.
- (iii) The red edges form a matching in  $G'$ .

From (i) and (ii), it follows by Theorem 3.1 that  $G'$  has a CAT  $T$ . From (iii),  $T$  must be an even alternating cycle. Then  $\Sigma(T)$  is a cycle in  $G$  with the required properties.  $\square$

#### 4. THE TRAIL CONE

Let  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$  be a 2-colored graph. In this section we show, using Theorem 3.1, that  $\mathcal{T}(G, \mathcal{C}) = \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ . Seymour [S] found the linear inequalities determining  $\mathcal{Z}(G)$  and our proof is modeled after his. Given a nonempty proper subset  $X$  of  $V$ , the subset  $D \subseteq E$  of edges between  $X$  and  $V - X$  will be called a *cut*. We say that  $X$  and  $V - X$  are the two *sides* of the cut, and their *sizes* are  $\#X$  and  $\#(V - X)$ . Let  $D$  be a cut,  $e \in D$ , and  $C$  a cycle in  $G$ . If  $C$  contains  $e$ , then  $C$  must also contain an edge in  $D - \{e\}$ . Thus the characteristic vector  $\chi(C)$  of  $C$  satisfies the following inequality

$$(4) \quad x(e) \leq \sum_{f \in D - e} x(f),$$

where we write  $D - e$  for  $D - \{e\}$ . We abbreviate the right-hand side of (4) by  $x(D - e)$ . We call (4) the *cut condition* for the pair  $(D, e)$ . If it holds with equality, the pair  $(D, e)$  is said to be *tight for  $x$* .

Seymour [S] proved the following result using Theorem 3.4.

**Theorem 4.1** (Seymour).  *$\mathcal{Z}(G)$  is the set of all  $x = (x(e) : e \in E)$  in  $\mathbb{R}^E$  satisfying the inequalities*

$$(5) \quad x(e) \leq x(D - e), \quad \text{for all cuts } D \text{ and all } e \in D,$$

$$(6) \quad x(e) \geq 0, \quad \text{for all } e \in E.$$

Vectors satisfying (5)–(6) are said to be *cut-admissible for  $G$* .

Given a graph  $G = (V, E)$ , we let  $\mathcal{K}(G)$  denote the set of all cycles in  $G$ . If  $\mathcal{C} : E \rightarrow \{R, B\}$  is a 2-coloring of  $G$ , then we denote the set of all CAT's in  $(G, \mathcal{C})$  by  $\mathcal{TR}(G, \mathcal{C})$ .

**Lemma 4.2.** *Let  $G = (V, E)$  be a graph and let  $p : E \rightarrow \mathbb{Q}^+$ . Let  $D$  be a cut in  $G$ , and let  $e \in D$  be such that  $(D, e)$  is tight for  $p$ .*

(i) *Suppose  $p \in \mathcal{Z}(G)$ , which means  $p$  can be expressed as*

$$p = \sum_{C \in \mathcal{K}(G)} \alpha(C) \chi(C), \quad \alpha(C) \in \mathbb{Q}^+.$$

*Let  $C \in \mathcal{K}(G)$  with  $\alpha(C) > 0$ . Then  $C \cap D$  is either empty or equal to  $\{e, h\}$  for some  $h \in D - e$  (we think of  $C$  as a set of edges).*

(ii) *Suppose that  $\mathcal{C} : E \rightarrow \{R, B\}$  is a 2-coloring and  $p \in \mathcal{T}(G, \mathcal{C})$ , which means  $p$  can be expressed as*

$$p = \sum_{T \in \mathcal{TR}(G, \mathcal{C})} \alpha(T) \chi(T), \quad \alpha(T) \in \mathbb{Q}^+.$$

*Let  $T \in \mathcal{TR}(G, \mathcal{C})$  with  $\alpha(T) > 0$ . Then  $T \cap D$  is either empty or equal to  $\{e, h\}$  for some  $h \in D - e$ .*

*Proof.* We prove (i); the proof of (ii) is similar. We have

$$\begin{aligned} \sum_{C \in \mathcal{K}(G)} \#(C \cap \{e\}) \alpha(C) &= \sum_{C \in \mathcal{K}(G), e \in C} \alpha(C) \\ &= p(e) \\ &= p(D - e) \\ &= \sum_{h \in D - e} \sum_{\substack{C \in \mathcal{K}(G) \\ h \in C}} \alpha(C) \\ &= \sum_{C \in \mathcal{K}(G)} \#(C \cap (D - e)) \alpha(C). \end{aligned}$$

Since each  $C \in \mathcal{K}(G)$  satisfies  $\#(C \cap \{e\}) \leq \#(C \cap (D - e))$ , it follows that each  $C \in \mathcal{K}(G)$  with  $\alpha(C) > 0$  satisfies  $\#(C \cap \{e\}) = \#(C \cap (D - e))$ . Since  $\#(C \cap \{e\}) \in \{0, 1\}$ , the result follows.  $\square$

**Lemma 4.3.** *Let  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$  be a 2-colored graph, let  $D$  be a cut in  $G$  with sides  $X$  and  $V - X$ , and let  $e \in D$  be an edge with an endpoint  $u_1 \in X$ . Let the pair  $(D, e)$  be tight for the weight function  $p : E \rightarrow \mathbb{Q}^+ - \{0\}$ .*

- (i) *Suppose that  $\#X = 1$  and that  $p$  satisfies the balance condition at the unique vertex of  $X$ . Then each edge of  $D - e$  has color opposite  $\mathcal{C}(e)$ . It follows that for each  $T \in \mathcal{TR}(G, \mathcal{C})$ , the intersection  $T \cap D$  is either empty or equal to  $\{e, h\}$  for some  $h \in D - e$ .*
- (ii) *Suppose that  $\#X = 2$  and that  $p$  satisfies the balance condition at both vertices of  $X$ . Then each edge in  $D - e$  with color opposite  $\mathcal{C}(e)$  has  $u_1$  as an endpoint, and each edge in  $D - e$  with color  $\mathcal{C}(e)$  does not have  $u_1$  as an endpoint. It follows that for each  $T \in \mathcal{TR}(G, \mathcal{C})$ , the intersection  $T \cap D$  is either empty or equal to  $\{e, h\}$  for some  $h \in D - e$ .*

*Proof.* (i) This is clear.

(ii) Let  $X = \{u_1, v_1\}$ . Without loss of generality we may assume that  $\mathcal{C}(e) = R$ . Set

$$\begin{aligned} x_1 &= p(e), \\ x_2 &= \sum_d p(d), \text{ where the sum is over all red edges } d \in D - e \text{ incident with } u_1, \\ x_3 &= \sum_d p(d), \text{ where the sum is over all blue edges } d \in D \text{ incident with } u_1, \\ x_4 &= \sum_d p(d), \text{ where the sum is over all red edges } d \in D \text{ incident with } v_1, \\ x_5 &= \sum_d p(d), \text{ where the sum is over all blue edges } d \in D \text{ incident with } v_1, \\ x_6 &= \sum_d p(d), \text{ where the sum is over all blue edges with both endpoints in } X, \\ x_7 &= \sum_d p(d), \text{ where the sum is over all red edges with both endpoints in } X. \end{aligned}$$

Since the pair  $(D, e)$  is tight we have

$$(7) \quad x_1 = x_2 + x_3 + x_4 + x_5.$$

The balance condition at  $u_1$  gives

$$(8) \quad x_1 + x_2 + x_7 = x_3 + x_6,$$

and the balance condition at  $v_1$  gives

$$(9) \quad x_5 + x_6 = x_4 + x_7.$$

Adding (8) and (9) gives

$$x_1 + x_2 + x_5 = x_3 + x_4.$$

Comparing this equation with (7) we get  $x_2 + x_5 = 0$ , and hence  $x_2 = x_5 = 0$  by nonnegativity. This implies that there are no red edges in  $D - e$  incident with  $u_1$  and no blue edges in  $D$  incident with  $v_1$ .  $\square$

For a graph  $G = (V, E)$  and a nonempty proper subset  $X$  of  $V$ , we denote by  $G'_X$  the graph obtained by shrinking  $X$  to a single vertex (and deleting the resulting loops).

**Lemma 4.4.** *Let  $G = (V, E)$  be a graph and let  $p : E \rightarrow \mathbb{Q}$ . Let  $X$  be a nonempty proper subset of  $V$  and let  $p'$  denote  $p$  restricted to the edges of  $G'_X$ . If  $p$  is cut-admissible for  $G$ , then  $p'$  is cut-admissible for  $G'_X$ .*

*Proof.* This follows since each cut in  $G'_X$  is also a cut in  $G$ .  $\square$

Let  $G = (V, E)$  be a graph with a 2-coloring  $\mathcal{C} : E \rightarrow \{R, B\}$ . For  $X, Y \subseteq V$ , we denote by  $\nabla_G(X, Y)$  the set of all edges of  $G$  with one endpoint in  $X$  and the other endpoint in  $Y$ . Let  $D$  be a cut in  $G$  with sides  $X$  and  $V - X$  of sizes at least 3, and let  $e \in D$  be an edge between  $u_1 \in X$  and  $u_2 \in V - X$ . Let  $p : E \rightarrow \mathbb{N} - \{0\}$  be a weight function.

Given these data, we define two edge-weighted 2-colored graphs  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ) by doing the following:

- Delete all edges in  $\nabla_G(X, X)$  (respectively,  $\nabla_G(V - X, V - X)$ ).
- Replace  $X$  (respectively,  $V - X$ ) with  $\{u_1, u'_1\}$  (respectively,  $\{u_2, u'_2\}$ ), where  $u'_1, u'_2 \notin V$  are two new vertices.
- The endpoints of each edge in  $\nabla_G(V - X, V - X) \cup \{e\}$  (respectively,  $\nabla_G(X, X) \cup \{e\}$ ) remain the same.
- The endpoint of each edge  $f \in D - e$  in  $V - X$  (respectively,  $X$ ) is the same as before, and the endpoint in  $X$  (respectively,  $V - X$ ) is  $u_1$  (respectively,  $u_2$ ) if  $\mathcal{C}(f) \neq \mathcal{C}(e)$  and is  $u'_1$  (respectively,  $u'_2$ ) if  $\mathcal{C}(f) = \mathcal{C}(e)$ .

- Add a new edge  $f_1$  (respectively,  $f_2$ ) between  $u_1$  and  $u'_1$  (respectively,  $u_2$  and  $u'_2$ ).
- The color of  $f_1$  (respectively,  $f_2$ ) is opposite  $\mathcal{C}(e)$ . All other edges retain their original color.
- Define a weight function  $p_1$  on the edges of  $G_X(e)$  by setting  $p_1(f_1) = \sum_h p(h)$ , where the sum is over all  $h \in D - e$  with  $\mathcal{C}(h) = \mathcal{C}(e)$ , and  $p_1(h) = p(h)$  for all other edges  $h$  of  $G_X(e)$ . Similarly, define a weight function  $p_2$  on the edges of  $G_{V-X}(e)$  by setting  $p_2(f_2) = \sum_h p(h)$ , where the sum is over all  $h \in D - e$  with  $\mathcal{C}(h) = \mathcal{C}(e)$ , and  $p_2(h) = p(h)$  for all other edges  $h$  of  $G_{V-X}(e)$ .

Our restriction on the sizes of  $X$  and  $V - X$  ensures that  $G_X(e)$  and  $G_{V-X}(e)$  have fewer vertices than  $G$ .

**Lemma 4.5.** *Let  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$  be a 2-colored graph, let  $D$  be a cut in  $G$  with sides  $X$  and  $V - X$  of sizes at least 3, and let  $e \in D$  be an edge between  $u_1 \in X$  and  $u_2 \in V - X$ . Let the pair  $(D, e)$  be tight for the weight function  $p : E \rightarrow \mathbb{N} - \{0\}$ .*

- (i) *Suppose that  $p$  satisfies the balance condition at each vertex of  $G$  and is cut-admissible for  $G$ . Then  $p_1$  (respectively,  $p_2$ ) satisfies the balance condition at each vertex of  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ) and is cut-admissible for  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ). Furthermore,  $(D, e)$  is tight for  $p_1$  and  $p_2$ .*
- (ii) *Suppose that  $p_1$  is a nonnegative integral combination of characteristic vectors of CAT's in  $G_X(e)$ , and  $p_2$  is a nonnegative integral combination of characteristic vectors of CAT's in  $G_{V-X}(e)$ . Then  $p$  is a nonnegative integral combination of characteristic vectors of CAT's in  $G$ .*

*Proof.* (i) From the definition of  $p_1(f_1)$  and  $p_2(f_2)$  and the hypothesis that  $(D, e)$  is tight for  $p$  and that  $p$  satisfies the balance condition at each vertex of  $G$ , it is clear that  $p_1$  and  $p_2$  satisfy the balance condition at each vertex of  $G_X(e)$  and  $G_{V-X}(e)$ , respectively, and that  $(D, e)$  is tight for  $p_1$  and  $p_2$ . We shall now verify that  $p_1$  is cut-admissible for  $G_X(e)$ ; the proof for  $p_2$  is the same.

Consider the graph  $G'_X$ . We retain the name  $u_1$  for the vertex obtained by shrinking  $X$ . Let  $p'$  be  $p$  restricted to the edges of  $G'_X$ . Note that  $D$  is a cut in  $G'_X$  and  $(D, e)$  is tight for  $p'$ . By Lemma 4.4,  $p'$  is cut-admissible for  $G'_X$  and thus, by Theorem 4.1, we can write

$$p' = \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(C'), \quad \alpha(C') \in \mathbb{Q}^+.$$

Consider a cycle  $C'$  in  $\mathcal{K}(G'_X)$  with  $\alpha(C') > 0$ . We obtain a cycle  $\overline{C'}$  in  $G_X(e)$  from  $C'$  as follows:

- If  $C'$  does not intersect  $D$ , then  $C'$  is a cycle in  $G_X(e)$  and we set  $\overline{C'} = C'$ .
- If  $C'$  intersects  $D$  then, by Lemma 4.2(i), this intersection must be  $\{e, h\}$  for some  $h \in D - e$ . If  $u_1$  is an endpoint of  $h$  in  $G_X(e)$  (i.e., if  $\mathcal{C}(e) \neq \mathcal{C}(h)$ ) then  $C'$  is also a cycle in  $G_X(e)$  and we set  $\overline{C'} = C'$ . If  $u_1$  is not an endpoint of  $h$  in  $G_X(e)$  (i.e., if  $\mathcal{C}(e) = \mathcal{C}(h)$ ) we define  $\overline{C'}$  to be the cycle in  $G_X(e)$  obtained from  $C'$  by inserting  $f_1$  between  $e$  and  $h$ .

We assert that

$$(10) \quad p_1 = \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'}).$$

To prove the assertion, we examine separately the two kinds of edges of  $G_X(e)$ : edges of  $G'_X$  and  $f_1$ . If  $f$  is an edge of  $G'_X$ , then each  $C' \in \mathcal{K}(G'_X)$  with  $\alpha(C') > 0$  satisfies  $\chi(C')(f) = \chi(\overline{C'})(f)$ , hence

$$\sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'})(f) = \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(C')(f) = p'(f) = p_1(f).$$

To prove the assertion, it remains to verify

$$p_1(f_1) = \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'})(f_1).$$

Consider any  $C' \in \mathcal{K}(G'_X)$  with  $\alpha(C') > 0$  and any  $h \in D - e$  with  $\mathcal{C}(h) = \mathcal{C}(e)$ . Then we have, by definition of  $\overline{C'}$ ,  $\chi(C')(h) = \chi(\overline{C'})(h) \leq \chi(\overline{C'})(f_1)$ . It follows that  $\chi(C')(h) \chi(\overline{C'})(f_1) = \chi(\overline{C'})(h)$ . Thus

$$(11) \quad \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(C')(h) \chi(\overline{C'})(f_1) = \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'})(h).$$

Now sum (11) over all  $h \in D - e$  such that  $\mathcal{C}(h) = \mathcal{C}(e)$ . Since for each  $C' \in \mathcal{K}(G'_X)$  such that  $\alpha(C') > 0$  and  $f_1 \in \overline{C'}$  there is exactly one edge  $h \in D - e$  such that  $\mathcal{C}(h) = \mathcal{C}(e)$  and  $h \in C'$ , the left-hand side of (11) sums to  $\sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'})(f_1)$ . The right-hand side of (11) sums to

$$\begin{aligned} \sum_{\substack{h \in D-e \\ \mathcal{C}(h)=\mathcal{C}(e)}} \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(\overline{C'})(h) &= \sum_{\substack{h \in D-e \\ \mathcal{C}(h)=\mathcal{C}(e)}} \sum_{C' \in \mathcal{K}(G'_X)} \alpha(C') \chi(C')(h) \\ &= \sum_{\substack{h \in D-e \\ \mathcal{C}(h)=\mathcal{C}(e)}} p'(h) \\ &= p_1(f_1). \end{aligned}$$

This proves the assertion (10). Thus  $p_1 \in \mathcal{Z}(G_X(e))$  and hence  $p_1$  is cut-admissible for  $G_X(e)$ .

(ii) The hypothesis on  $p_1$  (respectively,  $p_2$ ) implies that there is a multiset  $L_1$  (respectively,  $L_2$ ) of CAT's in  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ) such that every edge  $h$  in  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ) appears  $p_1(h)$  (respectively,  $p_2(h)$ ) times in the various CAT's contained in  $L_1$  (respectively,  $L_2$ ). We now build a multiset  $L$  of CAT's in  $G$  such that every edge  $h$  in  $G$  appears  $p(h)$  times in the CAT's contained in  $L$ . This will prove the result.

We begin with some notation. Let  $T$  be a CAT in  $G$  whose intersection with  $D$  has exactly 2 edges  $e$  and  $h$  for some  $h \in D - e$ . Let  $h_X$  and  $h_{V-X}$  be the endpoints of  $h$  in  $X$  and  $V - X$ , respectively. Then an appropriate cyclic shift of  $T$  must have the form

$$(u_2, e, u_1) * T_X * (h_X, h, h_{V-X}) * T_{V-X},$$

where  $T_X$  is a  $u_1$ - $h_X$  alternating trail whose vertices are in  $X$ , and  $T_{V-X}$  is a  $h_{V-X}$ - $u_2$  alternating trail whose vertices are in  $V - X$ .

Consider a CAT in  $L_1$  or  $L_2$  that intersects  $D$ . By Lemma 4.2(ii), the intersection of each such CAT with  $D$  must be  $\{e, h\}$  for some  $h \in D - e$ . For  $h \in D - e$ , let  $L_1(h)$  (respectively,  $L_2(h)$ ) consist of the CAT's in  $L_1$  (respectively,  $L_2$ ) whose intersection with  $D$  is  $\{e, h\}$ . By the definition of  $L_1$  and  $L_2$ ,  $p_1$  and  $p_2$ , we have  $\#L_1(h) = p_1(h) = p(h) = p_2(h) = \#L_2(h)$  for each  $h \in D - e$ . For each  $h \in D - e$ , fix a bijection  $\phi_h : L_1(h) \rightarrow L_2(h)$ .

We first take  $L$  to be empty and add CAT's to it as follows:

- Add to  $L$  all CAT's in  $L_1$  whose vertices are contained in  $V - X$  (each such CAT is added the same number of times as it appears in  $L_1$ ).
- Add to  $L$  all CAT's in  $L_2$  whose vertices are contained in  $X$ .
- For every  $h \in D - e$  and every  $T \in L_1(h)$  add the CAT

$$(u_2, e, u_1) * (\phi_h(T))_X * (h_X, h, h_{V-X}) * T_{V-X},$$

to  $L$ .

It is easily checked that each edge  $h$  in  $G$  appears  $p(h)$  times in the CAT's contained in  $L$  (in particular, this holds for  $h = e$  because  $(D, e)$  is tight for  $p$ ).  $\square$

**Theorem 4.6.** *Let  $G = (V, E)$ ,  $\mathcal{C} : E \rightarrow \{R, B\}$  be a 2-colored graph. Then*

$$\mathcal{T}(G, \mathcal{C}) = \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G).$$

*Proof.* We have already seen in the introduction that  $\mathcal{T}(G, \mathcal{C}) \subseteq \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$ . Consider a nonnegative rational vector  $q : E \rightarrow \mathbb{Q}^+$  that satisfies the balance condition at every vertex of  $G$  and that is cut-admissible for  $G$ . We will show that  $q \in \mathcal{T}(G, \mathcal{C})$ , which will prove the result. Without loss of generality we may assume that  $q(e) > 0$  for all  $e \in E$  (we may drop edges  $e$  with  $q(e) = 0$  and maintain the balance condition and cut-admissability). The proof is by induction on the pairs  $(\#V, \#E)$  ordered lexicographically.

The following two cases arise.

**Case (i):** there exist a cut  $D$  in  $G$  with sides  $X$  and  $V - X$  of sizes at least 3, and an edge  $e \in D$  such that  $(D, e)$  is tight for  $q$ .

For a suitably large positive integer  $N$ , the vector  $p = Nq$  is integral. Thus, by Lemma 4.5(i),  $p_1$  (respectively,  $p_2$ ) satisfies the balance condition at every vertex of  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ) and is cut-admissible for  $G_X(e)$  (respectively,  $G_{V-X}(e)$ ). Since  $G_X(e)$  and  $G_{V-X}(e)$  have fewer vertices than  $G$ , we see by induction and Lemma 4.5(ii) that for a suitably large positive integer  $M$ , the vector  $Mp$  is a nonnegative integral combination of characteristic vectors of CAT's in  $G$ . It follows that  $q \in \mathcal{T}(G, \mathcal{C})$ , as required.

**Case (ii):** for each cut  $D$  in  $G$  with sides  $X$  and  $V - X$  of sizes at least 3 and each  $e \in D$ , we have  $q(e) < q(D - e)$ .

Since  $q$  is positive on every edge and cut-admissible for  $G$ , it follows that  $G$  is bridgeless. Since  $q$  satisfies the balance condition at every vertex, it follows from Theorem 3.1 that  $(G, \mathcal{C})$  has a CAT  $T$ .

Consider the vector  $p_t = q - t\chi(T)$ ,  $t \geq 0$ . Clearly,  $p_t$  is balanced for all  $t \geq 0$  and nonnegative for all sufficiently small  $t > 0$ . We assert that there is a positive rational  $t_0$  such that  $p_{t_0}$  is cut-admissible for  $G$ . Indeed, let  $D$  be a cut in  $G$  with sides  $X$  and  $V - X$ , and let  $e \in D$ . We have the following two cases.

**Case (a):**  $q(e) < q(D - e)$ . Clearly  $p_t(e) < p_t(D - e)$  for all sufficiently small  $t > 0$ .

**Case (b):**  $q(e) = q(D - e)$ . By assumption, one of  $X$  and  $V - X$ , say  $X$ , has size at most 2. By Lemma 4.3 we see that either  $T$  contains no edge of  $D$  or it contains precisely two edges of  $D$ ,  $e$  and  $h$ , for some  $h \in D - e$ . It follows that  $p_t(e) = p_t(D - e)$  for all  $t \geq 0$ .

From these considerations we see that the maximum value of  $t$  such that

- $p_t(e) \geq 0$  for all  $e \in E$ ,
- $p_t$  is balanced,
- $p_t$  is cut-admissible for  $G$

is a positive finite rational  $t_0$ , as asserted. Set  $p = p_{t_0}$ . The following two subcases arise:

**Subcase (ii.1):**  $p(f) = 0$  for some  $f \in E$ . By dropping  $f$  we obtain a graph with the same number of vertices as  $G$  but with fewer edges, while maintaining balance and cut-admissability. Thus by induction  $p \in \mathcal{T}(G, \mathcal{C})$ , and hence  $q = p + t_0\chi(T) \in \mathcal{T}(G, \mathcal{C})$ , as required.

**Subcase (ii.2):**  $p(f) > 0$  for all  $f \in E$ . From case (b) above we see that  $q(e) = q(D - e)$  implies  $p(e) = p(D - e)$ . Since  $p$  is positive on every edge, it must be that the cutoff determining  $t_0$  occurs by case (a) above and not by case (b) or by the requirement that  $p_t \geq 0$ . Therefore there is a cut  $D^*$  and an edge  $e^* \in D^*$  such that  $q(e^*) < q(D^* - e^*)$  and  $p(e^*) = p(D^* - e^*)$ . Thus  $p$  is a positive rational vector, balanced and cut-admissible for  $G$ , and more pairs  $(D, e)$  are tight for  $p$  than for  $q$ . We may now repeat the whole argument with  $p$  in place of  $q$ . Since the total number of pairs  $(D, e)$  where  $D$  a cut in  $G$  and  $e \in D$  is finite, eventually we will reach case (i) or subcase (ii.1).  $\square$

Finally, we would like to state the following problems. In [BPS1] we saw that the problem of finding an integral vector in the intersection of the alternating cone with a box leads to the alternating



reachability problem. We can ask a similar question for the trail cone, but with the integrality restriction dropped. Given a 2-colored graph with nonnegative rational upper and lower bounds on the edges, is there an augmenting-path-type algorithm for either finding a rational vector in the trail cone satisfying these bounds, or showing that no such vector exists?

In [S] Seymour makes the following conjecture for a graph  $G = (V, E)$ : if  $y \in (2\mathbb{N})^E \cap \mathcal{Z}(G)$ , then  $y$  is a sum of cycles, i.e.,  $y$  can be written as a nonnegative integer linear combination of characteristic vectors of cycles in  $G$ . A vector in  $\mathcal{Z}(G)$  is a nonnegative rational combination of characteristic vectors of cycles, i.e., is a fractional sum of cycles. So Seymour's conjecture can be stated as follows: a fractional sum of cycles that is an even integer on every edge is a sum of cycles. Conjecture 1.1 states that a balanced sum of cycles is a sum of CAT's. Is there any relation between these conjectures?

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